MA2101 Linear Algebra II

Fields & Vector Spaces

- **Def 1.3 [Field, ring, group]**:
Closure, identity, associativity, invertibility, commutativity for $+$ and $\times$ (invertibility for $\times$ excludes 0), distributivity of $\times$ over $+$
Fields satisfy all axioms; (commutative) rings satisfy all except $\times$ invertibility; (additive) groups do not have $\times$ and may not satisfy $+$ commutativity
- **Def 1.12 [Vector space]**: Satisfies closure, identity, associativity, invertibility, distributivity, commutativity
- **Ex 1.18 [Matrix space]**: $M_n(F)$, $M_{m\times n}(F)$ — m rows

Vector Subspaces

- **Def 2.1 [Subspace]**: Set closed under vec. $+$ and scalar $\times$
- **Rem 2.3**: Subspace must have 0_W $(= 0_V)$
- **Thm 2.8 [Equiv. subspace def.]**: W is subspace of V
$\iff$ W is closed under linear combination
$\iff$ W with $+$ and $\times$ of V becomes a vector space
- **Ex 2.9**: Null space of matrix A is a subspace, i.e. solutions to homogeneous linear equations form a subspace
- **Thm 2.15**: The intersection of (possibly infinite number of) subspaces is a subspace: $\bigcap_{\alpha\in I} W_\alpha$ is a subspace
- **LA1 Ex 3.1.8.3 [2D & 3D geometry]**:
Line in $\mathbb{R}^2$: $\{(x,y) \mid ax+by=c\}$
Plane in $\mathbb{R}^3$: $\{(x,y,z) \mid ax+by+cz=d\}$
alternatively: (1) $\left\{\left(\frac{d-bs-ct}{a}, s, t\right) \mid s,t\in\mathbb{R}\right\}$ if $a\neq 0$
(2) $\left\{\left(s, \frac{d-as-ct}{b}, t\right) \mid s,t\in\mathbb{R}\right\}$ if $b\neq 0$
(3) $\left\{\left(s, t, \frac{d-as-bt}{c}\right) \mid s,t\in\mathbb{R}\right\}$ if $c\neq 0$
Line in $\mathbb{R}^3$: $\{(a_0,b_0,c_0)+t(a,b,c) \mid t\in\mathbb{R}\}$

Sums of Subspaces

- **Ex 3.3**: $S\subseteq T \implies \operatorname{Span}(S)\subseteq\operatorname{Span}(T)$
- **Thm 3.4**: S is subset of V, then $\operatorname{Span}(S)$ is subspace of V, and it is the smallest subspace containing S
- **Tut 2.5**: For subsets U_i, W_i, U, and subspace W:
(1) $U_1\subseteq U_2$ and $W_1\subseteq W_2 \implies U_1+W_1\subseteq U_2+W_2$
(2) $W+\{0\}=W$, $\quad W+W=W$
(3) $U+W=W \iff U\subseteq W$
- **Thm 3.11**: For subspaces $U,W\subseteq V$:
(1) $U+W=\operatorname{Span}(U\cup W)$
(2) $U+W$ is a subspace of V
(3) $U+W$ is the smallest subspace containing both U, W
- **Thm 3.18**: (induction of Thm 3.11) For subspaces $U_i\subseteq V$:
(1) $\sum_i W_i = \operatorname{Span}(\bigcup_i W_i)$
(2) $\sum_i W_i$ is a subspace of V
(3) $\sum_i W_i$ is the smallest subspace containing all W_i
- **Ex 3.15**: For subspaces $U,W\subseteq V$:
$U\cup W$ is subspace $\iff U\subseteq W$ or $W\subseteq U$
- **Ex 3.16**: $\operatorname{Span}(S\cup T)=\operatorname{Span}(S)+\operatorname{Span}(T)$
- **Def 3.20 [Direct sum of subspaces]**:
$W=W_1\oplus W_2 \iff W=W_1+W_2$ and $W1\cap W_2=\{0\}$
- **Thm 3.25 [Equiv. direct sum def.]**: W_i are subspaces of V, $W=W_1+W_2$:
W_1+W_2 is a direct sum $\iff \forall w\in W, w=w_1+w_2$ where $w_1\in W_1$, $w_2\in W_2$ and the w_1, w_2 are unique
- **Def 3.26 [Direct sum of many subspaces]**:
$$W=\bigoplus_i W_i \iff W=\sum_i W_i \text{ and } \forall k, \left(\sum_{i=1}^{k-1} W_i\right)\cap W_k=\{0\}$$
- **Thm 3.31 [Equiv. direct sum definition]**:
If W_i are subspaces of vector space V, $W=\sum_{i=1}^{s} W_i$ then:
$\sum_{i=1}^{s} W_i$ is a direct sum of W_i
$\iff \forall l, \left(\sum_{i\neq l} W_i\right)\cap W_l=\{0\}$
$\iff$ all w_i are unique in $w=\sum_{i=1}^{s} w_i$ (where $w_i\in W_i$)

Linear Independence

- **Def 4.1 [Linear independence]**: $S=\{v_i\}$ is a linearly independent set $\iff$ $(\sum_i x_i v_i = 0 \implies$ all $x_i=0)$
- **Thm 4.8 [Equiv. linear independence def.]**:
(1) $S=\{v_i\}$ is linearly independent
$\iff$ no $v_k\in S$ is a linear combination of others
(2) $S=\{v_1\}$ is linearly independent $\iff v_1\neq 0$
(3) $S=\{v_1,v_2\}$ is linearly independent
$\iff$ neither of v_1, v_2 is a scalar multiple of the other

Basis & Dimension

- **Def 4.10 [Basis]**: $B\subseteq V$ is a basis
$\iff V=\operatorname{Span}(B)$ and B is a linearly independent set
$\implies \dim_F V := |B|$ (for finite cardinality of B)
- **Thm 4.11 [Equiv. basis def. I]**: $B=v_i$ is a basis of V
$\iff \forall v\in V$, $v=\sum_i a_i v_i$ uniquely
$\iff V=\bigoplus_i \operatorname{Span}\{v_i\}$
- **Thm 4.13 [Basis from spanning set]**: $V=\operatorname{Span}(B)$
$\implies \exists B_1\subseteq B$ such that B_1 is a basis of V, and any maximal linear independent set $B_2\subseteq B$ is a basis of V
- **Thm 4.15 [Dimension is well-defined]**:
If B is a basis of V then:
(1) $|S|>|B| \implies S$ is linearly dependent
(2) $|T|<|B| \implies T$ does not span V
(3) B' is another basis of $V \implies |B'|=|B| (=\dim_F V)$
- **Thm 4.16 [Expanding a linearly independent set]**:
If $B\subseteq V$ is linearly independent then either:
(1) $\operatorname{Span}(B)=V$ so B is a basis of V
(2) $w\in V\setminus\operatorname{Span}(B) \implies B\cup\{w\}$ is linearly independent
Thus, $dim_F V=n \implies \exists w_{|B|+1},\dots,w_n\in V\setminus\operatorname{Span}(B)$ such that $B\coprod\{w_{|B|+1},\dots,w_n\}$ is a basis of V
- **Thm 4.18 [Equiv. basis def. II]**: $B=v_i$ is a basis of V
$\iff B$ is linearly independent and $|B|=\dim_F V$
$\iff \operatorname{Span}(B)=V$ and $|B|=\dim_F V$
- **Thm 4.19 [Basis and direct sum]**:
(1) $B=\coprod_i B_i$ is a basis of V
$\iff B_i$ is a basis of $W_i=\operatorname{Span}(B_i)$ and $V=\bigoplus_i W_i$
(2) $V=\bigoplus_i W_i \implies \dim_F V=\sum_i \dim_F W_i$
- **Ex 4.20**: If W is a subspace of (finite-dimensional) V then: $V=W \iff \dim V=\dim W$

Matrix Manipulation

- **LA1 Thm 2.2.22 [Transposition]**:
For any $A,B\in M_n(F)$ and $c\in F$:
(1) $(A^t)^t=A$
(2) $(A+B)^t=A^t+B^t$ (where A & B are the same size)
(3) $(cA)^t=cA^t$
(4) $(AB)^t=B^tA^t$ (where width of A = height of B)
- **LA1 Thm 2.2.22 [Inverse of square matrices]**:
For any invertible $A,B\in M_n(F)$ and $c\in F$:
(1) cA is invertible and $(cA)^{-1}=\frac{1}{c}A^{-1}$
(2) A^t is invertible and $(A^t)^{-1}=(A^{-1})^t$
(3) A^{-1} is invertible and $(A^{-1})^{-1}=A$
(4) AB is invertible and $(AB)^{-1}=B^{-1}A^{-1}$

Determinants & Invertibility
of a square matrix

- **LA1 Thm 2.5.6 [Cofactor expansions]**: $A\in M_n(F)$
$\det(A)=\sum_j a_{ij}(-1)^{i+j}\det(M_{ij})$, $\quad 1\le\forall i\le n$
$=\sum_i a_{ij}(-1)^{i+j}\det(M_{ij})$, $\quad 1\le\forall j\le n$
where M_{ij} is the (i,j)-minor of A
Note: $(-1)^{i+j}\det(M_{ij})$ is the (i,j)-cofactor of A
- **LA1 Thm 2.5.8 [Triangular matrix]**: The determinant of a triangular matrix is the product of its diagonal entries
- **LA1 Thm 2.5.10 [Transposition]**: $\det(A)=\det\left(A^t\right)$
- **LA1 Thm 2.5.19 [Invertibility]**:
A is invertible $\iff \det(A)\neq 0$
- **LA1 Thm 2.4.12 [Equiv. invertibility def.]**:
For any square matrix $A,B\in M_n(F)$, if $AB=I$ then:
(1) A and B are both invertible
(2) $A^{-1}=B$
(3) $B^{-1}=A$
(4) $BA=I$
- **LA1 Thm 2.5.22**: If $A,B\in M_n(F)$ and $c\in F$ then:
(1) $\det(cA)=c^n\det(A)$
(2) $\det(AB)=\det(A)\det(B)$
(3) $\det(A^{-1})=\dfrac{1}{\det(A)}$ if A is invertible
- **LA1 Thm 6.1.8 [Main theorem on invertible matrices]**: For any square matrix $A\in M_{n\times n}(F)$:
A is invertible
$\iff (AX=0 \implies X=0)$
$\iff \operatorname{rref}(A)=I$
$\iff A=\prod_i E_i$ where all E_i are elementary matrices
$\iff \det(A)\neq 0$
$\iff$ the rows of A form a basis for $\mathbb{R}^n$
$\iff$ the columns of A form a basis for $\mathbb{R}^n$
$\iff \operatorname{rank}(A)=n$
$\iff 0$ is not an eigenvalue of A
- **Thm 5.3 [Linear independence & determinant]**:
For any square matrix $A\in M_{n\times n}(F)$:
A is a invertible
$\iff \det(A)\neq 0$
$\iff$ The column vectors of A form a basis of F_c^n
$\iff$ The row vectors of A form a basis of F^n
$\iff$ The column vectors of A are linearly independent in F_c^n
$\iff$ The row vectors of A are linearly independent in F_c^n
$\iff (AX=0 \implies X=0)$, i.e. only has trivial solution

Row Spaces & Column Spaces

- **Def 5.1 [Basic definitions]**:
Column space: $\operatorname{Col}(A)$ is the span of column vectors in matrix A
Row space: $\operatorname{Row}(A)$ is the span of row vectors in matrix A
Range: $R(A):=\{AX \mid X\in F_c^n\}$
Null space (kernel): $\operatorname{Null}(A),\operatorname{Ker}(A):=\{X\in F_c^n \mid AX=0\}$
Nullity: $\operatorname{nullity}(A):=\dim\operatorname{Null}(A)$
Rank: *defined in Thm 5.2*
- **Thm 5.2 [Rank-dimension theorem]**:
(1) $R(A)=\operatorname{Col}(A)$
(2) $\dim\operatorname{Col}(A)=\dim\operatorname{Row}(A)$ $\quad(=:\operatorname{rank}(A))$
(3) $\operatorname{rank}(A)+\operatorname{nullity}(A)=$ number of columns in A
- **Ex 5.4**: For any $A\in M_{m\times n}(F)$:
$\operatorname{rank}(A)=$ Largest order of a minor of A that is invertible
(Note: A minor of A of order k is any matrix formed by removing $m-k$ rows and $n-k$ columns (so we are left with a $k\times k$ square matrix))

Row Equivalence

- **Def 5.5 [Elementary row operations]**:
Three types of elem. row ops. (representable by matrix):
(1) Switch row i with row j
(reflection, $B=EA \implies \det(B)=-\det(A)$)
(2) Add a scalar multiple of row i to row j
(shearing, $B=EA \implies \det(B)=\det(A)$)
(3) Multiply a nonzero scalar k to row i
(scaling, $B=EA \implies \det(B)=k\det(A)$)

If $B=E_s\cdots E_1A$ for some elementary row operations E_i, then matrices A and B are row equivalent
- **Thm 5.6 [Properties of row equivalent matrices]**:
If $A=(a_1,\cdots,a_n)$ and $B=(b_1,\cdots,b_n)$ are row equivalent (with a_i and b_i being column vectors), then:
(1) For any indices $i_1,\dots,i_s$ and $j_1,\dots,j_t$,
$c_1a_{i_1}+\cdots+c_sa_{i_s}=d_1a_{j_1}+\cdots+d_ta_{j_t}$
$\iff c_1b_{i_1}+\cdots+c_sb_{i_s}=d_1b_{j_1}+\cdots+d_tb_{j_t}$
(2) For any indices $i_1,\dots,i_s$,
$\{a_{i_1},\dots,a_{i_s}\}$ is a linearly independent set
$\iff \{b_{i_1},\dots,b_{i_s}\}$ is a linearly independent set
(3) For any indices $i_1,\dots,i_s$,
$\{a_{i_1},\dots,a_{i_s}\}$ form a basis for $\operatorname{Col}(A)$
$\iff \{b_{i_1},\dots,b_{i_s}\}$ form a basis for $\operatorname{Col}(B)$
(4) If B is in row-echelon form with leading entries at columns $i_1,\dots,i_s$ then
$\implies \{b_{i_1},\dots,b_{i_s}\}$ form a basis for $\operatorname{Col}(B)$
$\implies \{a_{i_1},\dots,a_{i_s}\}$ form a basis for $\operatorname{Col}(A)$ (by (3))
(5) $\operatorname{Row}(A)=\operatorname{Row}(B)$
(6) If B is in row-echelon form with leading entries at rows $1,\dots,t$ then
Rows $1,\dots,t$ of B form a basis for $\operatorname{Row}(B)=\operatorname{Row}(A)$
- **Ex 5.8 [Finding subset of vectors that form a basis]**:
Given some column vectors $v_1,\dots,v_n$, to find a subset that forms a basis for $\operatorname{Span}\{v_1,\dots,v_n\}$:
Find $A=\operatorname{rref}(v_1,\dots,v_n)$, and the column vectors corresponding to leading entries in A will form the basis
- **Ex 5.9 [Finding simpler basis for a set of vectors]**:
Given some row vectors $v_1,\dots,v_n$, to simpler basis for $\operatorname{Span}\{v_1,\dots,v_n\}$:
Find $A=\operatorname{rref}\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}$, and the nonzero row vectors in A will form the basis

- **Tut 4.4 [Finding rank and nullity of a matrix]**:
With $\operatorname{rref}(A)$, we can determine $\operatorname{rank}(A)$ from the number of leading '1's
$\operatorname{Null}(A)=\operatorname{Null}(\operatorname{rref}(A))$ may be determined like this:
$$R=\begin{pmatrix}1&0&-4&-28&-37&13\\0&1&-2&-12&-16&5\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\implies$$
$$\operatorname{Null}(R)=\left\{X=\begin{pmatrix}4x_3+28x_4+37x_5-13x_6\\2x_3+12x_4+16x_5-5x_6\\x_3\\x_4\\x_5\\x_6\end{pmatrix}\middle|\, x_i\in\mathbb{R}\right\}$$
- **Tut 3.3 [Checking polynomials for linear independence]**: To check if polynomials $u_1(x),u_2(x),u_3(x)$ are linearly independent, we need to form the equation $au_1(x)+bu_2(x)+cu_3(x)=0$ (identically) and group coefficients of the same type of term together, then solve the simultaneous equations that result
In general, we need to find a basis for the vector space

Quotient Spaces

- **Def 6.3 [Coset]**: $\overline{v}:=v+W:=\{v+w \mid w\in W\}$
- **Ex 6.4 [Equiv. coset relations]**:
$v+W=W$ (i.e. $\overline{v}=\overline{0}$)
$\iff v\in W$
$\iff v+W\subseteq W$
$\iff W\subseteq v+W$
- **Thm 6.5 [To be in the same coset]**:
$\overline{v_1}=\overline{v_2} \iff v_1-v_2\in W \iff v_2-v_1\in W$
- **Tut 1.4c [Equivalence classes are disjoint]**:
For any $v_1,v_2\in V$, either $\overline{v_1}\cap\overline{v_2}=\varnothing$ or $\overline{v_1}=\overline{v_2}$
- **Tut 1.4d [Cardinality]**: For any $v_1,v_2\in V$, $|\overline{v_1}|=|\overline{v_2}|$
- **Def 6.8 [Quotient space]**: If W is a subspace of V then:
$V/W:=\{\overline{v}=v+W \mid v\in V\}$, and operations on elements:
$(+)$ $\overline{v_1}+\overline{v_2}:=\overline{v_1+v_2}$
$(\times)$ $a\overline{v_1}:=\overline{av_1}$ $(a\in F)$
- **Def 6.9 [Quotient space is well-defined]**:
(1) The operations $+$ and $\times$ in Def 6.8 are well-defined
(2) V/W with $+$ and $\times$ in Def 6.8 becomes a vector space over F, with $0_{V/W}=\overline{0_V}=\overline{w}$ $(\forall w\in W)$
- **Rem 6.13 [Direct sum & quotient space]**:
If $V=U\oplus W$, then this map is an isomorphism:
$$U\to V/W$$
$$u\mapsto\overline{u}:=u+W$$
- **Ex 6.26 [Quotient map]**:
If W is a subspace of V, then this map is a surjective linear transformation called the quotient map:
$$\gamma\colon V\to V/W$$
$$v\mapsto\overline{v}:=v+W$$
In addition, $\operatorname{Ker}(\gamma)=W$

Linear Transformations

- **Def 6.10a [Linear transformation]**:
φ is a linear transformation from V_1 to V_2 $\iff$
$\varphi(v_1+v_2)=\varphi(v_1)+\varphi(v_2)$ and $\varphi(av_1)=a\varphi(v_1)$, $\forall v_i\in V$
A linear transformation from V to itself is a linear operator
- **Def 6.10b [Basic definitions]**: $\varphi\colon V\to W$
Domain: $\operatorname{dom}(\varphi):=V$
Codomain: $\operatorname{codomain}(\varphi):=W$
Kernel: $\operatorname{Ker}(\varphi):=\{v\in V \mid \varphi(v)=0_W\}$
Range (image): $\operatorname{R}(\varphi),\operatorname{Im}(\varphi):=\{\varphi(v) \mid v\in V\}\subseteq W$
- **Def 6.10c [Isomorphism]**:
An isomorphism is a bijective linear transformation
- **Ex 6.11 [Linear transformation T_A associated with a matrix A]**: $T_A\colon F_c^n\to F_c^m\colon X\mapsto AX$, then T_A is a linear transformation
$\operatorname{Ker}(T_A)=\operatorname{Ker}(A)=\operatorname{Null}(A)=\{X\in F_c^n \mid AX=0\}\subseteq F_c^n$
$\operatorname{R}(T_A)=T_A(F_c^n)=\operatorname{R}(A)=\{AX \mid X\in F_c^n\}\subseteq F_c^m$
- **Tut 4.5**: $T_A\colon F_c^n\to F_c^m\colon X\mapsto AX$
(1) T_A is injective $\iff \operatorname{nullity}(A)=0$ (i.e. $\operatorname{Null}(A)=\{0\}$)
(2) T_A is surjective $\iff \operatorname{rank}(A)=m$
(3) T_A is bijective (and hence an isomorphism) $\iff A$ is an invertible square matrix (i.e. $m=n$ and A^{-1} exists)
- **Rem 6.12**: For any linear transformation $T\colon V\to W$, $T(0_V)=0_W$
- **Thm 6.22 [Equiv. linear transformation def.]**:
φ is a linear transformation
$\iff \varphi(a_1v_1+a_2v_2)=a_1\varphi(v_1)+a_2\varphi(v_2)$ $\quad\forall a_i\in F, v_i\in V$
In particular: $\varphi(\operatorname{Span}(B))=\operatorname{Span}(\varphi(B))$
- **Ex 6.23**: If $\{u_i\}$ is a basis, $\{T(u_i)\}$ uniquely determines T
- **Ex 6.24 [Composition]**: $\varphi_2\varphi_1:=\varphi_2\circ\varphi_1$ is also a linear transformation
- **Thm 6.27 [Image is a subspace]**: If V_1 is a subspace (of V) then $T(V_1)=\{T(v) \mid v\in V_1\}$ is a subspace (of W)
- **Thm 6.28 [Kernel is a subspace]**:
(1) If φ is a linear transformation then $\operatorname{Ker}(\varphi)$ is a subspace
(2) If W is a subspace then there exists a linear transformation $\varphi\colon V\to U$ such that $\operatorname{Ker}(\varphi)=W$
- **Ex 6.29 [Inverse of a subspace is also a subspace]**:
If W_1 is a subspace (of W) then
$T^{-1}(W_1):=\{v\in V \mid T(v)\in W_1\}$ is a subspace (of V)
- **Ex 6.30 [Injectivity]**: φ is injective $\iff \operatorname{Ker}(\varphi)=\{0\}$
- **Ex 6.31a [Linear combinations]**:
For any linear transformations φ_1 and φ_2, these are all also linear tranformations:
(1) $\varphi_1+\varphi_2\colon v\mapsto\varphi_1(v)+\varphi_2(v)$
(2) $\alpha_1\varphi_1\colon v\mapsto\alpha_1\varphi_1(v)$
(3) $\alpha_1\varphi_1+\alpha_2\varphi_2\colon v\mapsto\alpha_1\varphi_1(v)+\alpha_2\varphi_2(v)$
- **Ex 6.31b**: The set of all linear transformations $\operatorname{Hom}_F(V,W):=\{\varphi\colon V\to W \mid \varphi$ is a linear transformation$\}$ is a vector space over F
- **Ex 6.32 [Powers of a linear operator]**:
Given any polynomial $f(x)=\sum_{i=0}^{n}a_ix^i$ and a linear operator T, $f(T):=\sum_{i=0}^{n}a_iT^i$ is also a linear operator
- **Ex 6.33 [$\operatorname{End}_F(V)$ is a ring]**: The set of all linear operators $\operatorname{End}_F(V):=\operatorname{Hom}_F(V,V)$ with the natural $+$ and $S\times T:=S\circ T$ is a (non-commutative) ring with $0_{End_F(V)}$ being the zero map and $1_{End_F(V)}$ being the identity map I_V

Isomorphisms

- **Ex 6.34**: $F_n\cong F_c^n$ $\qquad$ **Ex 6.35**: $M_{m\times n}(F)\cong F^{mn}$
- **Thm 6.37 [Isomorphism from kernel and image]**:
For any linear transformation $\varphi\colon V\to W$:
(1) φ is injective $\iff \operatorname{Ker}(\varphi)=\{0_V\}$
(2) φ is surjective $\iff \operatorname{Im}(\varphi)=W$
(3) φ is isomorphism $\iff \operatorname{Ker}(\varphi)=\{0_V\}$ and $\operatorname{Im}(\varphi)=W$
- **Thm 6.38 [1st isomorphism theorem]**:
For any linear transformation $\varphi\colon V\to U$, there exists an isomorphism
$$\overline{\varphi}\colon V/\operatorname{Ker}(\varphi)\xrightarrow{\sim}\operatorname{Im}(\varphi)\subseteq U$$
$$\overline{v}\mapsto\varphi(v)$$
such that $\varphi=\overline{\varphi}\circ\gamma$ where $\gamma\colon V\to V/\operatorname{Ker}(\varphi)\colon v\mapsto\overline{v}$
In particular:
φ is surjective $\iff \overline{\varphi}\colon V/\operatorname{Ker}(\varphi)\xrightarrow{\sim}U$ is an isomorphism
- **Thm 6.39 [Basis of a quotient space]**:
For any vector space V with finite dimension n, and subspace $W\subseteq V$ with a basis $B_W=\{v_1,\dots v_r\}$:
(1) By Thm 4.16, $\exists v_{r+1},\dots v_n$ such that $B_W\coprod\{v_{r+1},\dots,v_n\}$ is a basis of V
(2) $\{\overline{v_{r+1}},\dots,\overline{v_n}\}:=\{v_{r+1}+W,\dots,v_n+W\}$ is a basis of V/W $\quad(\therefore \dim_F V/W=\dim_F V-\dim_F W)$
(3) $B_W\coprod\{u_{r+1},\dots,u_n\}$ is a basis of V
$\iff \{\overline{u_{r+1}},\dots,\overline{u_n}\}$ is a basis of V/W
- **Rem 6.40**: Even if $\dim V=\infty$, it is still true that $\dim_F V=\dim_F W+\dim_F V/W$
- **Ex 6.41 [Properties of isomorphisms]**:
For any (possibly infinite-dimensional) isomorphism $\varphi\colon V\to W$ and subset $B\subseteq V$:
(1) $\sum_{i=1}^{r}a_iv_i=\sum_{i=r+1}^{s}a_iv_i \iff \sum_{i=1}^{r}a_i\varphi(v_i)=\sum_{i=r+1}^{s}a_i\varphi(v_i)$ for any $v_i\in V$ (i.e. $\varphi(v_i)\in W$)
(2) B is linearly independent $\iff \varphi(B)$ is linearly indep.
(3) $\varphi(\operatorname{Span}(B))=\operatorname{Span}(\varphi(B))$
(4) B spans $V \iff \varphi(B)$ spans W
(5) B is a basis of $V \iff \varphi(B)$ is a basis of W
(6) $\dim V=\dim W$
Note: (4) and (5) are true for any linear transformation
- **Ex 6.42 [Same-dimensional spaces are isomorphic]**:
For any finite-dimensional vector spaces V and W over F:
$\dim_F V=\dim_F W$
$\iff$ there exists an isomorphism $\varphi\colon V\xrightarrow{\sim}W$ (i.e. $V\cong W$)
$\iff$ for some n, $V\cong F^n\cong W$
- **Thm 6.43 [Dimension theorem]**:
For any linear transformation $\varphi\colon V\to W$ between (possibly infinite-dimensional) vector spaces over F:
$\dim_F\operatorname{Ker}(\varphi)+\dim_F\operatorname{Im}(\varphi)=\dim_F(V)$
- **Thm 6.44 [2nd isomorphism theorem]**:
For any subspaces W_1, W_2 of V, this map φ is a well-defined isomorphism
$$\varphi\colon W_1/(W_1\cap W_2)\xrightarrow{\sim}(W_1+W_2)/W_2$$
$$w+(W_1\cap W_2)=:\overline{w}\mapsto\overline{\overline{w}}:=w+W_2$$
Thus, $\dim W_1+\dim W_2=\dim(W_1+W_2)+\dim(W_1\cap W_2)$
- **Thm 6.45 [Equiv. isomorphism def. between same-dimensional spaces]**: For any linear transformation $\varphi\colon V\to W$ where $\dim V=\dim W<\infty$ (i.e. is finite):
φ is an isomorphism
$\iff \varphi$ is a surjection (i.e. $\operatorname{Ker}(\varphi)=\{0\}$)
$\iff \varphi$ is an injection (i.e. $\operatorname{Im}(\varphi)=W$)
- **Tut 6.7**: $T\colon V_1\to V_2$ is injective $\implies \dim V_1\le\dim V_2$
$T\colon V_1\to V_2$ is surjective $\implies \dim V_1\ge\dim V_2$
- *** [Equiv. bijectivity def.]**:
For any $f\colon V\to W$ and $g\colon W\to V$:
$g\circ f=\mathrm{id}_V$ and $f\circ g=\mathrm{id}_W \implies f$ and g are bijective and $f^{-1}=g$ and $g^{-1}=f$

Representation Matrices
of linear transformations

- **Ex 7.2**: For any $T\colon V\to W$, matrices C & D, and matrix of vectors $\mathbf{A}\in M_{m\times n}(V)$: $T(C\mathbf{A}D)=C(T(\mathbf{A}))D$
- **Def 7.4a $[[v]_B]$**: For a basis $B=(v_1,\dots v_n)$ and $v\in V$:
$[v]_B:=\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix}\in F_c^n$ such that $v=\sum_i c_iv_i$ (which is unique)
- **Def 7.4b [Recovering v from $[v]_B$]**: $v=B\,[v]_B$ $\quad$ (#2)
- **Ex 7.5 [Isomorphism of vectors]**:
$\varphi\colon V\to F_c^n\colon v\mapsto[v]_B$ is an isomorphism
- **Thm 7.6 [Representation matrix $[T]_{B,B_W}$]**:
For any $T\colon V\to W$ with $B=(v_1,\dots v_n)$ a basis of V and $B_W=(w_1,\dots w_n)$ a basis of W, there exists uniquely a matrix $[T]_{B,B_W}$ such that:
$[T(v)]_{B_W}=[T]_{B,B_W}[v]_B$ $\quad(\forall v\in V)$ $\quad$ (#3)
$[T]_{B,B_W}=\left([T(v_1)]_{B_W},\dots,[T(v_n)]_{B_W}\right)$ $\quad$ (#4)
$(T(v_1),\dots,T(v_n))=(w_1,\dots,w_m)\,[T]_{B,B_W}$ $\quad$ (#5)
- **Ex 7.8 [Isomorphism of transformations]**:
$\varphi\colon\operatorname{Hom}_F(V,W)\to M_{m\times n}(F)\colon T\mapsto[T]_{B,B_W}$ is an isom.
- **Ex 7.9 [Scalar transformations]**:
$T\colon v\mapsto\alpha v \iff [T]_B\colon v\mapsto\alpha v$ $\qquad(\forall\alpha\in F)$
- **Ex 7.10 [Finding $[T]_{B,B_W}$]**:
Express each $T(v_i)$ as a linear combination of $\{w_i\}$ using rref, and hence obtain $[T(v_i)]_{B_W}$, then use (#4)
- **Ex 7.13**: These are all isomorphisms $(A:=[T]_{B_V,B_W})$:
$\varphi\colon\ \operatorname{Ker}(T)\to\operatorname{Null}(A)\colon v\mapsto[v]_{B_V}$
$\psi\colon\ \operatorname{Null}(A)\to\operatorname{Ker}(T)\colon X\mapsto B_VX$
$\xi\colon R(T)\to R(T_A)\colon w\mapsto[w]_{B_W}$
$\eta\colon R(T_A)\to R(T)\colon Y\mapsto B_WY$
Consequently, T is an isomorphism $\iff A$ is invertible
- **Thm 7.14 [Represent. matrix for composite map]**:
$[T_2\circ T_1]_{B_1,B_3}=[T_2]_{B_2,B_3}[T_1]_{B_1,B_2}$
- **Ex 7.15 [Represent. matrix for inverse map]**:
$\left[T^{-1}\right]_{B_W,B_V}=\left([T]_{B_V,B_W}\right)^{-1}$
- **Ex 7.16 [Represent. matrix for linear combination]**:
$[a_1T_1+a_2T_2]_{B_V,B_W}=a_1[T_1]_{B_V,B_W}+a_2[T_2]_{B_V,B_W}$

Transition Matrices

- **Def 7.18 [Basic definition]**: If $B=(v_1,\dots,v_n)$ and $B'=(v_1',\dots,v_n')$ then the transition matrix from B' to B is $P_{B'\to B} \coloneqq P = ([v_1']_B,\dots,[v_n']_B) \in M_n(F)$
- **Thm 7.17 [Equiv. transition matrix def.]**: For any bases $B=(v_1,\dots,v_n)$ and $B'=(v_1',\dots,v_n')$ of V: $P=([v_1']_B,\dots,[v_n']_B) \iff B'=BP \iff P[v]_{B'}=[v]_B$
- **Thm 7.20 [Basis change theorem]**: $[T]_{B'} = P_{B\to B'}[T]_B P_{B'\to B} = (P_{B'\to B})^{-1}[T]_B P_{B'\to B}$
- **Def 7.23 [Similar matrices]**: $A_1\sim A_2 \iff A_2 = P^{-1}A_1P$ for some P
- **Ex 7.24**: Similarity '$\sim$' is an equivalence relation
- **Ex 7.25**: $A_1\sim A_2 \implies \det(A_1)=\det(A_2)$
- *** [Trace]**: Trace of matrix $\coloneqq$ product of diagonal elements
- **Def 7.26 [Determinant & trace of linear operator]**: $\det(T)\coloneqq\det([T]_B)$ and $\mathrm{Tr}(T)\coloneqq\mathrm{Tr}([T]_B)$ for any $T\colon V\to V$ and does not depend on choice of basis B
- **Def 7.27 [Characteristic polynomial]**: For $A\in M_n(F)$: $p_A(x)\coloneqq\det(xI_n-A)=x^n+b_{n-1}x^{n-1}+\dots+b_1x+b_0$ For $T\colon V\to V$, $p_T(x)\coloneqq p_{[T]_B}(x)$
- **Ex 7.28**: $A_1\sim A_2 \implies p_{A_1}(x)=p_{A_2}(x)$ thus $p_T(x)$ does not depend on choice of basis
- **Ex 7.29**: $\mathrm{Tr}(A)=-b_{n-1}$ $\quad(b_{n-1}\coloneqq$ coefficient of $x^{n-1})$
  $\det(A)=(-1)^n b_0$ $\quad(b_0\coloneqq$ coefficient of $x^0)$

Eigenvalues & Eigenvectors

- **Def 8.1 [Basic definitions]**: $T\colon V\to V$ with $T(v)=\lambda v$: v is an eigenvector corresponding to eigenvalue λ
- **Thm 8.3 [Equiv. eigenvalue & eigenvector def.]**:
  λ is eigenvalue of T corresponding to eigenvector v
  $\iff \lambda$ is eigenvalue of $[T]_B$ corresponding to eigenvector $[v]_B$
  $\iff \lambda I_V - T\colon V\to V : x\mapsto \lambda x - T(x)$ is not isomorphism
  $\iff \lambda I_n = [T]_B$ is not invertible
  $\iff \lambda$ is a root of $p_T(x)$ $\quad(=p_{[T]_B}(x))$
- **Ex 8.4 [Determinant]**: If $p_T(x)=(x-\lambda_1)\cdots(x-\lambda_n)$ where $\lambda_i\in\widetilde{F}$ then $\det(T)=\prod_{i=1}^n\lambda_i$
- **Def 8.5 [Eigenspace of an eigenvalue]**: For any eigenvalue λ: $V_\lambda(A)\coloneqq\mathrm{Ker}(\lambda I_n-A)=\{v\in V \mid T(v)=\lambda v\}$
- **Def 8.6 [Geometric & algebraic multiplicity of λ]**:
  Geometric multiplicity: $\dim V_\lambda$ $\qquad 1\le\dim V_\lambda\le n$
  Algebraic mult.: num. of repeated factors $(x-\lambda)$ in $p_T(x)$
  For any λ, (geometric mult. of λ) $\le$ (algebraic mult. of λ)
- **Ex 8.7 [Eigenspaces of T and $[T]_B$ are isomorphic]**: $f\colon \mathrm{Ker}(T-\lambda I_V)\to\mathrm{Null}([T]_B-\lambda I_n)\colon w\mapsto[w]_B$ is an isomorphism ($\therefore \dim V_\lambda(T)=\dim V_\lambda([T]_B)$)
- **Rem 8.8 [Bases of T and $[T]_B$]**:
  $\{u_1,\dots,u_s\}$ is a basis for $V_\lambda(T)$
  $\iff [u_1]_B,\dots,[u_s]_B$ is a basis for $V_\lambda([T]_B)$
  $\{X_1,\dots,X_s\}$ is a basis for $V_\lambda([T]_B)$
  $\iff \{BX_1,\dots,BX_s\}$ is a basis for $V_\lambda(T)$
- **Ex 8.9 [Eigenspaces of similar matrices]**: $P^{-1}AP=C \implies V_\lambda(A)=PV_\lambda(C)\coloneqq\{PX \mid X\in V_\lambda(C)\}$ ($\therefore \dim V_\lambda(A)=\dim V_\lambda(C)$)
- **Ex 8.11 [Finding eigenspaces of a matrix]**: Find $p_A(x)$ by diagonalizing (xI_3-A) and read off the eigenvalues, then for each λ_i compute $\mathrm{rref}(A-\lambda_iI_n)$ to find its null space $=V_{\lambda_i}(A)$
- **Ex 8.12 [Finding eigenspaces of a linear operator]**: Using a basis B (preferably the standard basis), find the matrix $[T]_B$, then use Ex 8.11 to compute $V_\lambda([T]_B)$ and Rem 8.8 to obtain $V_\lambda(T)$
- **Thm 8.14 [Sum of eigenspaces]**: For any eigenvalues $\{\lambda_1,\dots\lambda_k\}$, $\sum_{i=1}^k V_{\lambda_i}(T)$ is a direct sum
- **Tut 8.4a**: $Av=\lambda v \implies f(A)v=f(\lambda)v$ $\qquad(\forall f(x)\in F[x])$

T-invariant & T-cyclic Subspaces

- **Def 8.17 [T-invariant subspace]**: For $T\colon V\to V$: Subspace W (of V) is T-invariant $\iff T(W)\subseteq W$ then $T|W\colon W\to W\colon w\mapsto T(w)$ is the restriction of T on W
- **Ex 8.18**: $V_\lambda(T)$ and Span{<eigenvector>} are T-invariant
- **Ex 8.19**: W_1,W_2 are T-invar. $\implies W_1+W_2$ is T-invar.
- **Ex 8.23**: $T_1\circ T_2=T_2\circ T_1 \iff \mathrm{Ker}\,T_2, \mathrm{Im}\,T_2$ are T_1-invar.
- **Tut 7.6**: $T_1\circ T_2=T_2\circ T_1 \iff$ any e.sp. of T_2 is T_1-invar.
- **Ex 8.24**: W is T-invar. $\iff T(B_W)\subseteq W$ $\quad$(for basis B_W)
- **Def 8.15 [Polynomial $f(T)$]**: For $f(x)\coloneqq\sum_{i=0}^r a_ix^i$ then:
  $T\colon V\to V$ is lin. op. $\implies f(T)\coloneqq\sum_{i=0}^r a_iT^i$ is a lin. op.
  $A\in M_n(F) \implies f(A)\coloneqq\sum_{i=0}^r a_iA^i\in M_n(F)$
- **Ex 8.16 [Operations on $f(T)$]**: For $f(x),g(x)\in F[x]$:
  (1) $[f(T)]_B=f([T]_B)$ ($\therefore f(T)=0_{I_V} \iff f([T]_B)=0_{I_n}$)
  (2) $f(T)g(T)=f(T)\circ g(T)$
  (i.e. polynomial multiplication = composition)
  (3) $f(T)g(T)=g(T)f(T)$ $\qquad$(i.e. commutativity)
  (4) $f(P^{-1}AP)=P^{-1}f(A)P$ $\qquad$(for any invertible matrix P)
  (5) $f(S^{-1}TS)=S^{-1}f(T)S$ $\qquad$(for any isomorphism S)
- **Ex 8.25 [T-cyclic subspace]**: For $0\neq w_1\in V$
  (1) $W\coloneqq\mathrm{Span}\{T^s(w_1)\mid s\ge0\}$ is T-invariant
  (2) If $s>0$ is the smallest integer such that $T^s(w_1)\in\mathrm{Span}\{w_1,T(w_1),\dots,T^{s-1}(w_1)\}$ then: $\dim W=s$ and $\{w_1,T(w_1),\dots,T^{s-1}(w_1)\}$ is basis of W and $T^s(w_1)=\sum_{i=0}^{s-1}c_iT^i(w_1) \implies p_{T|W}(x)=\sum_{i=0}^{s-1}(-c_ix^i)+x^s$
  (can be proved using cofactor expansion or induction)
- **Thm 8.26**: $p_T(x)=q(x)p_{T|W}(x)$ for some $q(x)\in F[x]$
- **Ex 8.27 [Determinant of upper triangular matrix blocks]**: $\det\begin{pmatrix}C_1 & C_2\\ 0 & C_3\end{pmatrix}=\det(C_1)\det(C_3)$
- **Ex 8.28**: W is T-invariant $\iff$ $\quad B=(B_W,\ B_2)$
  $$[T]_B=\begin{pmatrix}A_1 & A_2\\ 0 & A_3\end{pmatrix}$$
  for square matrices A_1, A_3
  (i.e. the bottom-left cells are all zero (and $A_1=[T|W]_{B_W}$))
  Additionally, if $A_2=0$ then Span(B_2) is T-invariant too
- **Ex 8.29 [Upper triangular form of a transformation]**: If $p_T(x)=\prod_{i=1}^k(x-\lambda_i)^{n_i}$ (i.e. is fully factorizable) then there exists a basis B such that $[T]_B$ is upper triangular
  If $p_A(x)=\prod_{i=1}^k(x-\lambda_i)^{n_i}$ (i.e. is fully factorizable) then there exists an invertible P such that $P^{-1}AP$ is upper triangular
- **Ex 8.30 [Characteristic polynomial of direct sum]**: Given $T\colon V\to V$ and T-invariant subspaces W_i such that $V=\bigoplus_i W_i$, then $p_T(x)=\prod_i p_{T|W_i}(x)$
- **Ex 8.31 [Representation matrix of direct sum]**: $V=\bigoplus_i W_i$ (for T-invariant subspaces W_i) $\iff$ $[T]_B=\mathrm{diag}[A_1,\dots,A_r]$ (where size of $A_i=\dim W_i$) (when this is true, $A_i=[T|W_i]_{B_i}$ for some basis B_i of W_i)
- **Thm 8.31 [Cayley-Hamilton theorem]**:
  $p_T(T)=0I_V$ $\qquad p_A(A)=0I_n$

Minimal Polynomials

- **Def 9.1 [Minimal polynomial]**: $m_T(x)$ is the is the lowest-degree monic polynomial such that $m_T(T)=0I_V$
- **Rem 9.2**: $m_T(x)$ exists and $m_T(x)\mid p_T(x)$
- **Thm 9.3a**: $\quad f(T)=0I_V \iff m_T(x)\mid f(x)$
  $f(A)=0I_n \iff m_A(x)\mid f(x)$
- **Thm 9.3b [Uniqueness]**: Minimal polynomial is unique
- **Rem 9.19 [Same zero set as $p_T(x)$]**: $\{\alpha\in F \mid m_T(\alpha)=0\}=\{\alpha\in F \mid p_T(\alpha)=0\}$
- **Ex 9.4 [Finding $m_A(x)$]**: Simply check all factors of $p_A(x)$ which have at least degree 1 of each eigenvalue
- **Ex 9.5 [Sim. matrices]**: $A_1\sim A_2 \implies m_{A_1}(x)=m_{A_2}(x)$
- **Ex 9.6 [Direct sum]**: $A=\mathrm{diag}[A_1,\dots,A_r]$ $\implies m_A(x)=\mathrm{lcm}\{m_{A_1}(x),\dots,m_{A_r}(x)\}$
  Equivalently, $V=\bigoplus_i W_i$ and W_i are T-invariant $\implies m_T(x)=\mathrm{lcm}_i\{m_{T|W_i}(x)\}$

Jordan Canonical Form

- **Def 9.8 [Jordan block]**: $J_s(\lambda)\coloneqq\begin{pmatrix}\lambda & 1 & 0 & \cdots & 0 & 0\\ 0 & \lambda & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \lambda & 1\\ 0 & 0 & 0 & \cdots & 0 & \lambda\end{pmatrix}$
  $J_s(\lambda)\in M_s(F)$
  (1) $m_{J_s}(\lambda)=p_{J_s}(\lambda)=(x-\lambda)^s$
  (2) $V_\lambda(J_s(\lambda))=\mathrm{Span}\left\{(1,0,\cdots,0)^t\right\}$
  (3) geometric mult. of $\lambda=1$, algebraic mult. of $\lambda=s$
- **Ex 9.9 [Jordan canonical form with eigenvalue $\boldsymbol{\lambda}$]**: If $A=\mathrm{diag}_i[J_{s_i}(\lambda)]$ then:
  (1) $m_A(x)=\mathrm{lcm}_i\{m_{J_{s_i}}(\lambda)\}$ (= max Jordan block size)
  (2) $V_\lambda(A)$ = span of columns corresponding to the first element in each Jordan block (these columns form a basis)
  (3) geom. mult. of $\lambda=\dim V_\lambda(A)$ = num. of Jordan blocks
- **Def 9.11 [Jordan canonical form]**: $J=\mathrm{diag}[A(\lambda_1),\dots,A(\lambda_k)]$ where $A(\lambda_i)$ are from Ex 9.9 is a Jordan canonical form
  $p_J(x)=\prod_{i=1}^k p_{A(\lambda_1)}(x)=\prod_{i=1}^k(x-\lambda_i)^{(\text{size of }A(\lambda_i))}$
  $m_J(x)=\prod_{i=1}^k m_{A(\lambda_1)}(x)=\prod_{i=1}^k(x-\lambda_i)^{(\text{max Jordan block in }A(\lambda_i))}$
- **Thm 9.14 [Linear operator]**: If $p_T(x)$ is fully factorizable, i.e. $p_T(x)=\prod_i(x-\lambda_i)^{n_i}$ and $m_T(x)=\prod_i(x-\lambda_i)^{m_i}$, then there exists a basis B such that $J\coloneqq[T]_B$ is in Jordan canonical form, and J is unique up to re-ordering of Jordan blocks
- **Thm 9.15 [Matrix]**: If $p_A(x)$ is fully factorizable, i.e. $p_A(x)=\prod_i(x-\lambda_i)^{n_i}$ and $m_A(x)=\prod_i(x-\lambda_i)^{m_i}$, then there exists an invertible P such that $J\coloneqq P^{-1}AP$ is in Jordan canonical form, and J is unique up to re-ordering of Jordan blocks
- **Thm 9.16 [T and $[T]_B$]**: For any basis $\widetilde{B}$ of T and Jordan canonical form J: $P^{-1}[T]_BP=J \iff [T]_{(\widetilde{B}P)}=J$
- **Thm 9.18 [Factorizability in F]**:
  A (or T) has a Jordan canonical form
  $\iff p_A(x)$ (or $p_T(x)$) is fully factorizable in F
- **Thm 9.20**: $A_1\sim A_2 \iff A_1$ and A_2 have the same Jordan canonical form (after reordering of Jordan blocks)
- **Ex 10.1 [Solving differential equations]**: $y_i=y_i(x)$ Solve simult. eqns. $y_i'=a_{i1}y_1+a_{i2}y_2+a_{i3}y_3$ for $i=1,2,3$:
  $Y\coloneqq\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}, Y'\coloneqq\begin{pmatrix}y_1'\\y_2'\\y_3'\end{pmatrix}, A\coloneqq\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}$
  Find a Jordan canon. form $J\coloneqq P^{-1}AP$, and $Z\coloneqq P^{-1}Y$
  Then $PZ'=(PZ)'=Y'=AY=APZ$, i.e. $Z'=JZ$
  ...which can be solved by substitution since J is triangular

Diagonalizability

- **Def 9.21 [Basic definition]**:
  T is diagonalizable $\iff \exists$ basis B s.t. $[T]_B$ is diagonal
  A is diagonalizable $\iff \exists$ invertible matrix P s.t. $P^{-1}AP$ is diagonal
  In particular, if $B=(v_1,\dots,v_n)$ and $P=(p_1,\dots,p_n)$ then:
  $T(v_i)=\lambda_iv_i$ (where $[T]_B=\mathrm{diag}[\lambda_1,\dots,\lambda_n]$)
  $Ap_i=\lambda_ip_i$ (where $P^{-1}AP=\mathrm{diag}[\lambda_1,\dots,\lambda_n]$)
  i.e. B is an eigenbasis for P
  Given eigenvectors and corresponding eigenvalues, T and A can be re-constructed
- **Thm 9.23 [Equiv. diagonalizable def.]**:
  T is diagonalizable over F
  $\iff [T]_{B'}$ is diagonalizable over F for some basis B'
  $\iff [T]_{B'}$ is diagonalizable over F for all bases B'
  $\iff$ a basis B can be formed from (some) eigenvectors of T
  $\iff$ there exists n linearly independent eigenvectors of T
  $\iff$ for some (and hence every) basis B', a basis P for $[T]_{B'}$ can be formed from (some) eigenvectors of $[T]_{B'}$
  $\iff$ for some (and hence every) basis B', there exists n linearly independent eigenvectors of $[T]_{B'}$
  $\iff$ If $\lambda_{m_1},\dots,\lambda_{m_k}$ are the only distinct eigenvalues of T, and B_i is a basis of $V_{\lambda_{m_i}}(T)$, and $B=(B_1,\dots,B_k)$, then $[T]_B=\mathrm{diag}\left[\lambda_{m_1}I_{|B_1|},\dots,\lambda_{m_k}I_{|B_k|}\right]$
  In above def., $B=(v_1,\dots,v_n) \iff P=([v_1]_{B'},\dots,[v_n]_{B'})$
- **Thm 9.24 [Diagonalizable def. for $\boldsymbol{m_T(x)}$ & $\boldsymbol{m_A(x)}$]**: The equivalences applies to both T and A:
  T is diagonalizable over F
  $\iff m_T(x)$ is a product of distinct linear polynomials in F (i.e. $m_T(x)=\prod_i(x-\lambda_i)$ where all λ_i are distinct)
  $\iff m_T(x)$ is fully factorizable in F with no repeated roots
  $\iff$ If $p_T(x)=\prod_i(x-\lambda_i)^{n_i}$ where all λ_i are distinct, then $\dim V_{\lambda_i}=n_i$ (i.e. geom. mult = alg. mult.)
- **Rem 9.25 [Finding P to diagonalize A]**: Find $p_A(x)=\prod_i(x-\lambda_i)^{n_i}$, ensure $m_A(x)$ has no repeated roots (i.e. is diagonalizable), then take any basis B_i of eigenspace $V_{\lambda_i}(A)$, then $P=(B_1,\dots B_k)\in M_n(F)$ will make $P^{-1}AP$ diagonalizable
- **Tut 7.7 [Multiplication with upp. triangular vanishing diagonal matrices]**: For $r,s\ge0$:
  $A=(a_{ij})\in M_n(F)$ with $a_{ij}=0$ $(\forall j-i\le r)$ and
  $B=(b_{ij})\in M_n(F)$ with $b_{ij}=0$ $(\forall j-i\le s)$
  $\implies C\coloneqq AB=(c_{ij})$ has $c_{ij}=0$ $(\forall j-i\le r+s+1)$
  In particular, $A^n=0$
- **Ex 9.29a [Nilpotence]**: T is nilpotent $\iff \exists m\in\mathbb{Z}^+$ such that $T^m=0I_V$
- **Ex 9.29b [Equiv. nilpotence def.]**: T has dimension n:
  T is nilpotent
  $\iff$ every eigenvalue of T is zero
  $\iff p_T(x)=x^n$
  $\iff m_T(x)=x^s$ for some $s\in\mathbb{Z}^+$
- **Ex 9.29c [Semi-simplicity & nilpotence]**: Semi-simple = diagonalizable
  If T has a Jordan canonical form, then there exists unique semi-simple T_s and nilpotent T_n such that $T=T_s+T_n$
  This T_s and T_n satisfy:
  (1) $T_s\circ T_n=T_n\circ T_s$
  (2) There are polynomials $f(x)$ and $g(x)$ such that $T_s=f(T)$ and $T_n=g(T)$
  (3) For any linear operator S, $T\circ S=S\circ T \implies T_s\circ S=S\circ T_s$ and $T_n\circ S=S\circ T_n$
- **Tut 8.4b**: A is diagonalizable $\implies f(A)$ is diagonalizable

Bilinear Forms

- **Def 10.2a [Basic definition]**: $H\colon V\times V\to F$ is a bilinear form
  $\iff \begin{cases}H(a_1x_1+a_2x_2,y)=a_1H(x_1,y)+a_2H(x_2,y)\\ H(x,b_1y_1+b_2y_2)=b_1H(x,y_1)+b_2H(x,y_2)\end{cases}$
  $(\forall x,x_1,x_2,y,y_1,y_2\in V,\forall a_1,a_2,b_1,b_2\in F)$
- **Def 10.2b [Symmetry]**: H is symmetric $\iff H(x,y)=H(y,x)$ $(\forall x,y\in V)$
- **Ex 10.3a [Bilin. form $\boldsymbol{H_A}$ associated with matrix A]**: For a basis $B=(v_1,\dots,v_n)$ and matrix $A=(a_{ij})\in M_n(F)$:
  $H_A\colon V\times V\to F\colon \left(\sum_{i=1}^n x_iv_i,\sum_{j=1}^n y_jv_j\right)\mapsto\sum_{i=1}^n\sum_{j=1}^n a_{ij}x_iy_j$
  ...is a bilinear form $(=X^tAY$ where $X=\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}, Y=\begin{pmatrix}y_1\\\vdots\\y_n\end{pmatrix})$
- **Ex 10.3b [Representation matrix]**: Given a basis $B=(v_1,\dots,v_n)$, H_A is the bilinear form associated with A, and this association is bijective
- **Ex 10.3c [Symmetry of $\boldsymbol{H_A}$]**: H_A is symmetric $\iff A$ is symmetric (i.e. $A^t=A$)
- **Def 10.5 [Non-degenerate]**: H is non-degenerate $\iff$ $(H(x,y_0)=0\ (\forall x\in V) \implies y_0=0)$
  H_A is non-degenerate $\iff A$ is invertible (for one/any basis)

Inner Products

- **Def 10.10a [Inner product $\langle\,,\rangle$]**: $H\colon V\times V\to\mathbb{R}$ is a real inner product if:
  (1) H is a bilinear form
  (2) H is symmetric
  (3) $x\neq0 \implies \langle x,x\rangle>0$
  $H\colon V\times V\to\mathbb{C}$ is a complex inner product if:
  (1) H is linear about x and conjugate linear about y, i.e.:
  $$\langle a_1x_1+a_2x_2,y\rangle=a_1\langle x_1,y\rangle+a_2\langle x_2,y\rangle$$
  $$\langle x,b_1y_1+b_2y_2\rangle=\overline{b_1}\langle x,y_1\rangle+\overline{b_2}\langle x,y_2\rangle$$
  $$(\forall x,x_1,x_2,y,y_1,y_2\in V,\forall a_1,a_2,b_1,b_2\in\mathbb{C})$$
  (2) H is conjugate symmetric, i.e. $\langle y,x\rangle=\overline{\langle x,y\rangle}$
  (3) $x\neq0 \implies \langle x,x\rangle>0$
- **Def 10.10b [Norm]**: $\|x\|\coloneqq\sqrt{\langle x,x\rangle}\ge0$
  $\|x\|=0 \iff x=0_V$
- **Def 10.10c [Orthogonal]**: $x\perp y \iff \langle x,y\rangle=0$
- **Ex 10.11 [Non-degeneracy of inner product]**:
  $\langle u_0,y\rangle=0$ $(\forall y\in V) \implies u_0=0_V$
  $\langle x,v_0\rangle=0$ $(\forall x\in V) \implies v_0=0_V$
- **Def 10.12 [Orthonormal basis]**: $B=(v_1,\dots,v_n)$ is an orthonormal basis (relative to $\langle\,,\rangle$) if:
  (1) $v_i\perp v_j$ $(\forall i\neq j)$
  (2) $\|v_i\|=1$ $(\forall i)$
- **Ex 10.13a [Standard inner product]**:
  $H\colon V\times V\to\mathbb{R}\colon(X,Y)\mapsto X^tY$ $\qquad$(real version)
  $H\colon V\times V\to\mathbb{C}\colon(X,Y)\mapsto X^t\overline{Y}$ $\qquad$(complex version)
  ($\overline{X}$ = element-wise complex conjugate of X)
- **Ex 10.13b [Standard basis is orthonormal]**: Any permutation of the standard basis is an orthonormal basis (relative to the standard inner product)
- **Ex 10.13c [Inner product subspace]**: If W is subsp. of V and (V,H) is inner product space then: $(W,H|W)$ is an inner product space
- **Ex 10.15 [Gram-Schmidt process]**:
  $\tilde{v}_1\coloneqq u_1 \quad \tilde{v}_2\coloneqq u_2-\frac{\langle u_2,\tilde{v}_1\rangle}{\|\tilde{v}_1\|^2}\tilde{v}_1 \quad\cdots\qquad v_j\coloneqq\frac{\tilde{v}_j}{\|\tilde{v}_j\|}$

Adjunction

with an inner product space $F=\mathbb{R}$ or $\mathbb{C}$

- **Def 10.16 [Adjoint matrix]**: $A^*\coloneqq\left(\overline{A}\right)^t=\overline{(A^t)}$
- **Ex 10.17 [Standard inner product]**: $\langle AX,Y\rangle=\langle X,A^*Y\rangle$ $(\forall A\in M_n(F),\forall X,Y\in V)$
- **Thm 10.18 [Adjoint linear operator $\boldsymbol{T^*}$]**:
  (1) Given $T\colon V\to V$, there exists unique $T^*\colon V\to V$ s.t.: $\langle T(u),v\rangle=\langle u,T^*(v)\rangle$ $(\forall u,v\in V)$
  (2) Given any orthonormal basis B: $\quad[T^*]_B=([T]_B)^*$
- **Ex 10.19 [Adjoint of adjoint]**: $(T^*)^*=T$
  $\therefore\langle T^*(u),v\rangle=\langle u,T(v)\rangle$ $(\forall u,v\in V)$
- **Ex 10.20 [Identities of adjunction]**:
  (1) $T=\alpha I_V \implies T^*=\overline{\alpha}I_V$ $(\forall\alpha\in\mathbb{C})$
  (2) $(a_1T_1+a_2T_2)^*=\overline{a_1}T_1^*+\overline{a_2}T_2^*$
  (3) $(T_1\circ T_2)^*=T_2^*\circ T_1^*$
- **Def 10.21 [Unitary and self-adjoint linear operators]**:
  (1) $T\colon\mathbb{R}\to\mathbb{R}$ is orthogonal $\iff TT^*=I_V$ (i.e. $T^*T=I_V$)
  (2) $T\colon\mathbb{C}\to\mathbb{C}$ is unitary $\iff TT^*=I_V$ (i.e. $T^*T=I_V$)
  (3) $T\colon\mathbb{R}\to\mathbb{R}$ is symmetric $\iff T^*=T$
  (4) $T\colon\mathbb{C}\to\mathbb{C}$ is self-adjoint (or Hermitian) $\iff T^*=T$
  (5) T is normal $\iff TT^*=T^*T$
  Similar defs. exist for $A\in M_n(F)$ replacing $T\colon F\to F$
  $\therefore T$ is unitary or self-adjoint $\implies T$ is normal
- **Ex 10.22 [Linear operators & rep. matrices]**: For any basis B of T:
  T is unitary $\iff [T]_B$ is unitary
  T is self-adjoint $\iff [T]_B$ is self-adjoint
  T is normal $\iff [T]_B$ is normal
  Similar equivalences exist for orthogonal or symmetric T
  $\therefore [T]_B$ is unitary (or self-adj. or normal)
  $\implies [T]_{B'}$ is unitary (or self-adj. or normal) $(\forall B')$
- **Thm 10.23/10.24 [Equiv. orthogonal/unitary matrix def.]**: $F=\mathbb{R}$ (for orthogonal) or $\mathbb{C}$ (for unitary)
  For any matrix $P\in M_n(F)$, using standard inner product:
  P is orthogonal/unitary (i.e. $PP^*=I_n$)
  $\iff P=(p_1,\dots,p_n)$ forms an orthonormal basis for F_c^n
  $\iff \forall X,Y\in F_c^n:\quad\|PX-PY\|=\|X-Y\|$
  $\iff \forall X\in F_c^n:\quad\|PX\|=\|X\|$
  $\iff$ for one (and hence every) orthonormal basis B of F_c^n, $B'\coloneqq BP$ is an orthonormal basis of F_c^n
- **Ex 10.25 [Eigenvalues of unitary matrices]**: If $P\in M_n(\mathbb{C})$ is unitary then $|\lambda|=1$ for all eigenvalues λ ($\therefore|\det(P)|=1$)
  If $P\in M_n(\mathbb{R})$ is orthogonal then $|\lambda|=1$ for all (possibly complex) eigenvalues λ, and $\det(P)=\pm1$
- **Ex 10.26 [Eigenvalues of self-adjoint matrices]**:
  (1) $A\in M_n(\mathbb{C})$ is self-adjoint $\implies p_A(x)$ has all real roots
  (2) $T\colon\mathbb{C}\to\mathbb{C}$ is self-adjoint $\implies p_T(x)$ has all real roots
  (3) T (or A) is self-adjoint $\implies$ for any eigenvectors v_1,v_2 corresponding to distinct eigenvalues, $v_1\perp v_2$
- **Def 10.27a [Positive/negative definite linear ops.]**:
  T is positive definite $\iff T$ is self-adjoint and $\langle T(v),v\rangle>0$ $(\forall0\neq v\in V)$
  T is negative definite $\iff T$ is self-adjoint and $\langle T(v),v\rangle<0$ $(\forall0\neq v\in V)$
  $\therefore T$ is negative definite $\iff -T$ is positive definite
- **Def 10.27b [Positive/negative definite matrices]**:
  A is positive definite $\iff A$ is self-adjoint and $(AX)^t\,\overline{X}=X^tA^t\overline{X}>0$ $(\forall0\neq X\in F_c^n)$
  T is negative definite $\iff T$ is self-adjoint and $(AX)^t\,\overline{X}=X^tA^t\overline{X}<0$ $(\forall0\neq X\in F_c^n)$
  $\therefore A$ is negative definite $\iff -A$ is positive definite
- **Rem 10.28a [Leading principal minors]**: If $A\in M_n(\mathbb{R})$ is symmetric then:
  A is positive definite $\iff$ all leading principal minors of A have positive determinants
  (leading principle minors = upper-left square sub-matrices)
- **Rem 10.28b [Equiv. positive definite def.]**: If T is self-adjoint (on an inner product space over $\mathbb{R}$ or $\mathbb{C}$) with orthonormal basis B, and $A\coloneqq[T]_B$:
  T is positive definite
  $\iff A$ is positive definite
  $\iff$ every eigenvalue of T is positive
  $\iff$ every eigenvalue of A is positive
  $\iff A=C^*C$ for some invertible matrix $C\in M_n(\mathbb{C})$
- **Ex 10.30** $H\colon\mathbb{C}_c^n\times\mathbb{C}_c^n\to\mathbb{C}:(X,Y)\mapsto(AX)^t\overline{Y}$ is an inner product $\iff A$ is positive definite
- **Tut 10.6a [Normal op. properties]**: If T is normal then:
  (1) $\|T(v)\|=\|T^*(v)\|$ $\quad(\forall v\in V)$
  (2) $T=\alpha T_V$ is normal $\quad(\forall\alpha\in\mathbb{C})$
  (3) $T(v)=\lambda v \implies T^*(v)=\overline{\lambda}v$
  (4) v_1,v_2 are eig.vectors of distinct eigenvalues $\implies v_1\perp v_2$
- **Tut 10.6b [Schur's theorem]**:
  (1) $T^*(w)=\lambda w \implies (\mathrm{Span}\{w\})^\perp$ is T-invariant
  (Thm) $\exists$ orthonormal basis B s.t. $[T]_B$ is upper triangular

Principal Axis Theorem

- **Def 10.8 [Congruence]**: Matrices $A,B\in M_n(F)$ are congruent $\iff B=P^tAP$ for some $P\in M_n(F)$
  Congruence is an equivalence relation
  Representation matrix of bilinear form is congruent under change of basis: If $X=PY$ then: $H(PY_1,PY_2)=H(X_1,X_2)=X_1^tAX_2=(PY_1)^tA(PY_2)=Y_1^t(P^tAP)Y_2$
- **Thm 10.31 [Principal axis theorem]**:
  (1) T (on real inner prod. space) is symmetric (= self-adj.) $\iff\exists$ orthonormal basis B such that $[T]_B\in M_n(\mathbb{R})$ is diag.
  (2) $A\in M_n(\mathbb{R})$ is symmetric (= self-adj.) $\iff\exists$ orthogonal matrix P such that $P^{-1}AP(=P^tAP)\in M_n(\mathbb{R})$ is diagonal
  (3) T (on complex inner product space) is normal $\iff$ $\exists$ orthonormal basis B such that $[T]_B$ is diagonal
  (4) $A\in M_n(\mathbb{C})$ is normal $\iff\exists$ unitary matrix U such that $U^{-1}AU(=U^*AU)\in M_n(\mathbb{C})$ is diagonal
- **Ex 10.32 [Orthogonal complement]**: Let W be a subspace of V, and B_W be an orthonormal basis of W, and $W^\perp\coloneqq\{x\in V\mid\langle x,w\rangle=0,\forall w\in W\}$
  (1) B_W can be extd. to an orthonormal basis $B=(B_W,B_2)$ of V (just extend then orthonormalize via Gram-Schmidt process), and any such B_2 is an orthonormal basis of $W^\perp$
  (2) $V=W\oplus W^\perp$

Quadratic Forms

- **Def 10.33a [Quadratic forms]**: $K\colon V\to F$ is a quadratic form $\iff\exists$ symmetric bilin. form H s.t. $K(x)\equiv H(x,x)$
  When $V=F_c^n$, all quadratic forms are of the form
  $K\colon F_c^n\to F:\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}=X\mapsto X^tAX=\sum_{i=1}^n\sum_{j=1}^n a_{ij}x_ix_j$
  for some symmetric A
- **Def 10.33b**: Every quadratic form on F_c^n is a homogeneous polynomial of degree 2, and vice-versa (choose $a_{ij}=a_{ji}=\frac{1}{2}$(coef. of x_ix_j) to make A symmetric)
- **Thm 10.34 [Quadratic form of principal axis thm.]**:
  For any quadratic form $f(x_1,\dots,x_n)=\sum_{i=1}^n\sum_{j=1}^n a_{ij}x_ix_j$,
  $\exists$ orthogonal $P\in M_n(\mathbb{R})$ s.t. $f(x_1,\dots,x_n)=\sum_{i=1}^n\lambda_iy_i^2$
  (*"standard form"*) where $\begin{pmatrix}y_1\\\vdots\\y_n\end{pmatrix}=P^{-1}\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}$ and
  $\lambda_1,\dots,\lambda_n\in\mathbb{R}$ are the eigenvalues of the symmetric matrix $A=(a_{ij})\in M_n(\mathbb{R})$
  This expression is unique up to relabelling of λ_iy_i
- **Rem 10.35 [Finding $\boldsymbol{P}$ to diagonalize a symmetric (real) matrix $\boldsymbol{A}$]**: Get all eigenvalues & eigenspaces of A, then get an orthonormal basis for each eigenspace using Gram-Schmidt process, then concatenate the bases for form P *(since eigenspaces of distinct eigenvalues are orthogonal for symmetric matrices)* and $P^tAP=\mathrm{diag}[\text{<eigenvalues>}]$
- **Tut 10.4a [Cauchy-Schwarz ineq.]**: $|\langle x,y\rangle|\le\|x\|\,\|y\|$
- **Tut 10.4b [Triangle ineq.]**: $\|x+y\|\le\|x\|+\|y\|$